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t-Designs and t-ply Homogeneous Groups

RICHARD N. LANE

Systems Applications, Inc., Beverly Hills, Calif.

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ABSTRACT

Permutation groups transitive on unordered sets of t points are studied in relation to block designs in which sets of t points occur equally often in blocks. Many such designs can be created from such groups, and the normal structure of a design admitting one of these groups as automorphisms is found to be quite regular and to depend upon a subgroup structure of the group.

In this paper we perform synthesis and analysis of t-designs which admit an automorphism group transitive on the set of unordered sets of t points of the design. Such groups as general permutation groups are known as t-ply homogeneous (sometimes t-homogeneous) groups and are reviewed in Section 1. Section 2 shows how any t-ply homogeneous group can be used to construct a variety of t-designs on the permuted points. The basic parameters of t-designs constructed in this way can be derived from various group relationships. In Section 3 some constructions of t-designs from known groups are given. In this process a new characterization of the linear fractional groups LF(3, q) arises. We also construct an apparently new infinite class of 3-designs from the triply transitive groups PGL(2, q). The remainder of the paper is devoted to the normal structure of t-designs admitting a t-ply homogeneous automorphism group. The basic theory of the normal structure of t-designs was developed in [5], but the relevant facts are reviewed here in Section 4. Section 5 applies this theory and the results of Section 2 to the class of designs here, and relates the normal design structure to a corresponding subgroup structure in the t-ply homogeneous group.

The formal definition of a t-design used here is a pair $T = \langle B, \Pi \rangle$, where Π is a set of points and B is a set of subsets (blocks) of Π, with the two properties: (1) each element of B contains k elements of Π for some fixed integer k, and (2) each set of t-elements of Π lies in λ elements of B for some fixed integer $\lambda > 0$. Other parameters are b for the number of blocks, v for the number of points, and r for the number of blocks in which each point occurs.

The notations used are essentially those of Wielandt [6]: sets are represented by capital Greek letters, functions and permutations by lower case Greek letters, points by numerals or lower case Latin letters. Structures, such as groups or designs, are represented by capital Latin letters. If G is a permutation group on the set Ω, 1, 2,..., t points of Ω, and Γ a subset of Ω, then $G_{1,2,\ldots,t}$, G_Γ, and $G_{\Gamma,1,2,\ldots,t}$ are, respectively, the subgroups fixing the t points 1, 2,..., t; the subgroup sending the set Γ into itself (not necessarily fixing it pointwise), and the subgroup fixing the points 1, 2,..., t *and* sending the set Γ into itself. If Ξ is a collection of orbits of the group G, then G^Ξ is the permutation representation of G on Ξ. If Φ is any set or group, $|\Phi|$ means the number of elements in that set or group. If T is a t-design, we write $G(T)$ for the automorphism group of T. We say that G is transitive, primitive, etc., if G has that property when represented as a permutation group on $\Pi(T)$.

1. t-PLY HOMOGENEOUS GROUPS

Let G be a group of permutations of the set Ω. Then, for $\Phi \subseteq \Omega$ and $\alpha \in G$, we write

$$\Phi^\alpha = \{x^\alpha \mid x \in \Phi\},$$

so G acts as a permutation group on the unordered k-sets of Ω for any $k \leqslant |\Omega| = n$. If some union of orbits of the representation of G on k-sets forms a t-design then G is by definition an automorphism group of that design.

DEFINITION 1.1. G is called *t-ply homogeneous* if for any two t-sets of Ω, say Φ_1 and Φ_2, there exists an $\alpha \in G$ such that $\Phi_1{}^\alpha = \Phi_2$. The subgroup sending Φ into itself is called G_Φ.

Clearly any t-ply transitive group is t-ply homogeneous, but groups do exist which are t-ply homogeneous but not t-ply transitive, for example, the group on seven letters generated by

$$x = (abcdefg),$$

$$y = (a)(bce)(dgf),$$

is 2-ply homogeneous but not 2-ply transitive. Such groups are called strictly t-ply homogeneous.

t-ply homogeneous groups have been studied for some time, e.g., in [1]. D. R. Hughes [4] has shown that a t-ply homogeneous group is $(t-1)$-ply transitive if the group is of sufficiently large degree. A relation between t-ply homogeneous groups and t-ply transitive groups is given by

PROPOSITION 1.2. Let G be t-ply homogeneous on the set Ω. Then G is t-ply transitive if and only if for some t-set $B \subset \Omega$, $G_B{}^B \approx S^B$, the symmetric group on B.

PROOF. The necessity of the condition is obvious. To show its sufficiency, let $B = \{b_i\}_{i=1}^t$. We shall show that, for any ordered t-set $C = \{c_i\}_{i=1}^t$, there is an $\alpha \in G$ such that $b_i^\alpha = c_i$ for $1 \leqslant i \leqslant t$. Since G is t-ply homogeneous, there is a $\beta \in G$ such that $B^\beta = C$. Let $b_{i'} = c_i^{\beta^{-1}}$. By hypothesis, then, there is a $\gamma \in G_B$ such that $b_i^\gamma = b_{i'}$ and if α is set equal to $\gamma\beta$, $b_i^\alpha = c_i$ for $1 \leqslant i \leqslant t$, and G is t-ply transitive.

For $t = 2$, the condition is readily applied:

COROLLARY 1.3. *A 2-ply homogeneous group is 2-ply transitive if and only if it has even order.*

PROOF. If G has even order, there is an involution $\alpha \in G$:

$$\alpha = (a, b) \cdots$$

and then

$$G_{\{a,b\}}^{\{a,b\}} = \{I, (a,b)\} = S^{\{a,b\}},$$

so G is doubly transitive.

Conversely, if G is doubly transitive of degree n, the even number $n(n-1)$ divides the order of G, since it is the index of the stabilizer of two points.

PROPOSITION 1.4. If the group G of degree n is strictly 2-ply homogeneous on Ω, then

(i) G has rank 3,

(ii) G is primitive,

(iii) G_1 has orbits of length 1, $(n-1)/2$, $(n-1)/2$,

(iv) $n \equiv 3(4)$,

(v) G is solvable,

(vi) n is a prime power p^r,

(vii) G contains a regular normal minimal elementary Abelian subgroup N.

Conversely, any rank 3 group of odd order is strictly 2-ply homogeneous.

PROOF. (i) If i and j are any two distinct points of Ω and a is a third point, there exists $\alpha \in G : \{a, i\}^\alpha = \{a, j\}$. If $a^\alpha = a, i^\alpha = j$, set $\beta = \alpha$. If $a^\alpha = j$, $i^\alpha = a$, set $\beta = \alpha^2$. Then β takes i into j and G is transitive. If $1 \in \Omega$, we compute the orbits of G_1. For $2 \in \Omega - \{1\}$, set $\Gamma = 2^{G_1}$. $\Gamma \neq \Omega - \{1\}$, because G is not doubly transitive, so we can pick a $j \in \Omega - \{1\} - \Gamma$. If $\Gamma \cup \{1\} \cup \{j\} = \Omega$, G_1 has those three sets as orbits so G is of rank 3. Otherwise, there is a further point k in $\Omega - \{1, j\} - \Gamma$. We wish to show the existence of an element fixing 1 and carrying j into k, so G_1 will have the three orbits $1, \Gamma, j^{G_1}$.

$k \notin \Gamma$, so there is no element of form

$$\begin{pmatrix} 1 & 2,\dots \\ 1 & k,\dots \end{pmatrix}$$

in G, hence an element taking $(1, 2)$ into $(1, k)$ must be of form

$$\alpha_1 = \begin{pmatrix} 1 & 2,\dots \\ k & 1,\dots \end{pmatrix}.$$

Arguing identically on j, we obtain the element

$$\alpha_2 = \begin{pmatrix} 1 & 2,\dots \\ j & 1,\dots \end{pmatrix}$$

in G, and

$$\alpha_2^{-1}\alpha_1 = \begin{pmatrix} 1 & j,\dots \\ 1 & k,\dots \end{pmatrix}$$

is in G_1, hence G has rank 3.

(ii) A rank 3 group of odd order is primitive (Higman [3]).

(iii) G is transitive on unordered pairs. There are $n(n-1)/2$ of these. n and this number must divide G, hence both are odd, and $n \equiv 3 \pmod 4$.

(v) Feit-Thompson.

(vi) 11.5 of Wielandt [6].

(vii) 11.5 of Wielandt [6].

Conversely, suppose G is a rank 3 group of odd order. 16.5 of Wielandt implies that the lengths of the orbits of G_0 are $1, (n-1)/2, (n-1)/2$ (where n is the degree of G). 3.2 of Wielandt implies that the length of the orbit of $\{0, 1\}$ by G is $[G : G_{\{0,1\}}]$. There is no element of G interchanging 0 and 1, so $G_{\{0,1\}} = G_{0,1}$, hence

$$[G : G_{\{0,1\}}] = [G : G_{0,1}] = [G : G_0] \cdot [G_0 : G_{0,1}] = n \cdot (n-1)/2,$$

whichever orbit of G_0 1 lies in. Hence G carries $\{0, 1\}$ into $n(n-1)/2$ different unordered pairs. Since there are only $n(n-1)/2$ unordered pairs in all, G is 2-ply homogeneous, and strictly so because $|G|$ is odd.

There are many such groups, for example: Let G be the sharply doubly transitive group of linear substitutions $z \to az + b (a \neq 0)$ in a near-field K, where K is of order $p^r \equiv 3(4)$. Then G has order $p^r(p^r - 1) \equiv 2(4)$. This is twice an odd number, so G has a subgroup G^* of index 2, and G^* is strictly (sharply) 2-ply homogeneous on the points of K.

2. t-designs with t-ply Homogeneous Groups

THEOREM 2.1. *Let G be a t-ply homogeneous permutation group on Ω. Then for any $\Phi \subseteq \Omega$, the pair*

$$T = \langle \{\Phi^\alpha\}_{\alpha \in G'}, \Omega \rangle$$

is a t-design admitting G as an automorphism group with parameters

$$v = |\Omega|, \qquad b = [G : G_\Phi], \qquad k = |\Phi|,$$

$$r = \frac{k}{v}[G : G_\Phi], \qquad \lambda = \binom{k}{t} \frac{|G_{\{1,2,\ldots,t\}}|}{|G_\Phi|}.$$

Conversely, any t-design T' admitting G as an automorphism group is a union of designs of this form.

PROOF. By definition of G as a permutation group, $\{\Phi^\alpha\}$ is a set of k-subsets of Ω. $[G : G_\Phi]$ is the length of the orbit of Φ under G, hence the number of distinct sets Φ^α is $[G : G_\Phi]$.

Let A be a t-set of Ω. We wish to count the number of distinct sets Φ^α such that $A \subseteq \Phi^\alpha$. For any such set Φ^α, $(\Phi^\alpha)^\beta = \Phi^\alpha$ for any $\beta \in G_\Phi \alpha = \alpha^{-1} G_\Phi \alpha$, so there are $|G_\Phi|$ group elements for each such set Φ^α.

We now count the number of group elements α such that $A \subseteq \Phi^\alpha$. But for each such α, $A^{\alpha^{-1}} \subseteq \Phi$, so we may count the number of group elements α such that $A^\alpha \subseteq \Phi$. But there are $\binom{k}{t}$ t-sets of Φ, and the image of A can be any one of them. Furthermore, for each possible image, there are $| G_A | = | G_{\{1,2,\ldots,t\}} |$ group elements sending A into that image. Hence there are

$$\binom{k}{t} \cdot | G_{\{1,2,\ldots,t\}} |$$

group elements α in all with $A \subseteq \Phi^\alpha$, or

$$\binom{k}{t} \frac{| G_{\{1,2,\ldots,t\}} |}{| G_\Phi |}$$

such distinct sets Φ^α. Thus T is a t-design, and the value for r follows by similar computation.

For the converse, suppose G has orbits $\Gamma_1, \ldots, \Gamma_u$ on the blocks of T'. Picking arbitrary blocks $\Phi_1, \Phi_2, \ldots, \Phi_u$ such that $\Phi_i \in \Gamma_i$ for $1 \leqslant i \leqslant u$, the first statement of the theorem shows that $T_i = \langle \Gamma_i , \Omega \rangle$ must be a subdesign of T', and, since $\Gamma_i \cap \Gamma_i = \phi$ if $i \neq j$, $T' = \langle \cup_i \Gamma_i , \Omega \rangle$ by construction.

The t-design constructed in Theorem 2.1 is called the *action* of G on Φ, written Φ^G. Theorem 2.1 has many applications to t-designs admitting such groups of automorphisms. For the remainder of this section, let T be a t-design, and let G be a t-ply homogeneous group of automorphisms of T.

One immediate and useful result is

THEOREM 2.2. *If $\lambda(T) = 1$ and Φ is any block, then $G_\Phi{}^\Phi$ is t-ply homogeneous.*

PROOF. Let A and B be any two t-sets of Φ. There is an $\alpha \in G$ such that $A^\alpha = B$. Alpha must then send all blocks containing A into blocks containing B. But, since $\lambda = 1$, Φ is the only block containing A, also the only block containing B. Therefore α must send Φ into Φ, so $\alpha \in G_\Phi$, which is then t-ply homogeneous.

3. SOME CONSTRUCTIONS

By Theorem 2.1, many t-designs can be constructed from a given t-ply homogeneous group, but many of these will be trivial or uninteresting, because the parameters b and λ are exceedingly large by comparison with

v and k, respectively. For example, the Mathieu group M_{12} is quintuply transitive on 12 letters, and it is not difficult to show that M_{12} is t-ply homogeneous for all t between 1 and 12, except for $t = 6$. Thus $\Phi^{M_{12}}$ will be the trivial design of all $|\Phi|$-sets, unless $|\Phi| = 6$, in which case $\Phi^{M_{12}}$ is one of the well-known few 5-designs. Inspection of the formulas for b and λ in Theorem 2.1 yields the information that a design of the form Φ^G will only have reasonable parameters if G_Φ is quite large in G. In the extreme case of Φ^G being a symmetric design, we must have $|G_\Phi| = |G_0|$. Such subgroups appear to be quite scarce. For example, we have

THEOREM 3.1. *Let G be a 2-ply transitive permutation group on Ω, and suppose G has a subgroup H such that H is 2-ply transitive on an orbit Γ, and if* 0, 1 *are any two points of Γ, $|H| = |G_0|$ and $G_{0,1} \subset H$. Then $v = |\Omega| = p^{2r} + p^r + 1$ for some prime p and some integer*

$$r > 0, k = |\Gamma| = p^r + 1,$$

and G is isomorphic to a subgroup of $LF(3, p^r)$.

PROOF. We apply Theorem 2.1 to Γ^G. Since

$$H \subseteq G_\Gamma, b(\Gamma^G) = [G : G_\Gamma] \leqslant [G : H] = v, \text{ so } b \leqslant v.$$

Also, from 2.1,

$$\lambda = \binom{k}{2} \frac{|G_{\{0,1\}}|}{|G_\Gamma|} = \frac{k^2 - k}{2} \cdot \frac{2 \cdot |G_{0,1}|}{|G_\Gamma|}.$$

G_Γ is surely doubly transitive on Γ, so

$$\lambda = \frac{(k^2 - k)|G_{0,1}|}{(k^2 - k)|G_{0,1,\Gamma}|} \leqslant \frac{|H_{0,1}|}{|H_{0,1,\Gamma}|} = 1.$$

Since $\lambda > 0, \lambda = 1, G_\Gamma = H, b = v$, and Γ^G is a symmetric 2-design with $\lambda = 1$, otherwise known as a projective plane. Since the collineation (automorphism) group of Γ^G contains the doubly transitive (on points) group G, Γ^G must be Desarguesian and can be coordinatized by a finite field with p^r elements for some prime p and integer $r > 0$. The group of such a plane is LF$(3, p^r)$, and so G is by construction isomorphic to a subgroup of LF$(3, p^r)$.

Some interesting 2-designs (balanced incomplete block designs), can be constructed from known groups. A doubly primitive group is one which is doubly transitive and whose subgroups fixing a point are primitive on the remaining points.

THEOREM 3.2. *Let G be doubly transitive but not doubly primitive, and let Γ be a non-trivial block of G_0. Then Γ^G is a block design (2-design) with $\lambda \mid k - 1$.*

PROOF: Let 1 be a point of Γ. It is well known (see, e.g., [2, pp. 64–65]) that $G_{0,\Gamma}$ is transitive on Γ, and that $G_{0,1} \subseteq G_{1,\Gamma}$. Hence the group $G_{0,\Gamma}$ of order $k \cdot | G_{0,1} |$ is a subgroup of G_Γ, G_Γ is transitive on Γ, and from 2.1,

$$\lambda = \binom{k}{2} \frac{| G_{\{0,1\}} |}{| G_\Gamma |} = \frac{k(k-1)| G_{0,1} |}{| G_\Gamma |} = \frac{(k-1)| G_{0,1} |}{| G_{\Gamma,1} |} = \frac{k-1}{\left(\frac{G_{\Gamma,1}}{G_{0,1}}\right)}.$$

Since $G_{0,1} \subset G_{\Gamma,1}$, the denominator is an integer, and $\lambda \mid k - 1$.

It is also known that the projective linear groups PGL$(2, p^n)$ of distinct linear fractional transforms $z \to (\alpha z + \beta)/(\gamma z + \delta)(\alpha, \beta, \gamma, \delta$ in GF(p^n), $\alpha\delta - \beta\gamma \neq 0)$ is triply transitive on the $p^n + 1$ symbols consisting of the elements of GF(p^n) plus the symbol ∞. An infinite class of 3-designs can be constructed from these groups in the following way. If $n > 1$, and s is a proper division of n, then GF(p^n) has a proper subfield Σ isomorphic to GF(p^s). Furthermore, the subgroup of PGL$(2, p^n)$ sending the set $\Gamma = \{\infty\} \cup \Sigma$ into itself is isomorphic to PGL$(2, p^s)$. If we set $G =$ PGL$(2, p^n)$, then Γ^G is a 3-design with parameters

$$v = p^n + 1, \qquad k = p^s + 1, \qquad \lambda = 1,$$

$$b = p^{n-s} \frac{(p^n + 1)(p^n - 1)}{(p^s + 1)(p^s - 1)}, \qquad r = p^{n-s} \frac{(p^n - 1)}{(p^s - 1)},$$

as can be derived from direct computation and Theorem 2.1.

If $n/s = 2$, the design is easily seen to be an inversive plane, and for other values of n/s similar geometric interpretations exist.

4. SKETCH OF BASIC NORMAL STRUCTURE THEORY

The normal structure of a t-design is similar to that of a group. Analogous to the chief series of a group is the chief series for a t-design T:

$$T = T_0 \rhd T_1 \rhd \cdots \rhd T_2,$$

where each $T_i (i > 0)$ is a maximal normal subdesign of T_{i-1} and the quotient designs T_{i-1}/T_i are simple. Here a normal subdesign of the t-design $T = \langle B, \Pi \rangle$ is a t-design $T' = \langle B', \Pi' \rangle$ where

(1) $B' \subset B, \Pi' \subset \Pi, \lambda(T') = \lambda(T)$,

and

(2) $B(T)$ can be partitioned into the block sets of a set $\{T' = T_1, T_2, T_2, \ldots, T_m\}$ of subdesigns (codesigns of T') of T with the same parameters b, v, k, r, λ as T'.

(A complete subdesign is one which satisfies only condition 1.) If $T' \lhd T$, then the quotient design T/T' is the pair $\langle \{B(T_i)\}_{i=1}^{m}, \Pi \rangle$ and is a t-design with parameters

$$b = b(T)/b(T'), \qquad v = v(T), \qquad r = r(T)/r(T'),$$
$$k = v(T), \qquad \lambda = \lambda(T)/\lambda(T')$$

There are several interesting types of homomorphic maps from one t-design to another, but the one of relevance here is the regular block homomorphism, which in every case is equivalent to a map $\alpha : T_1 \to T_2$, where T_2 is a quotient design T_1/T' for some $T' \lhd T_1$. The map itself sends each point onto itself and each block onto the point set of the codesign to which the block belongs. A t-design is simple if it has no non-trivial normal subdesigns, or equivalently, no non-trivial regular block homomorphic images.

For further details and additional results, please see [5].

5. Normal Structures Continued

The normal structure of t-designs with t-ply homogeneous groups is readily determined. The following hold for any t-design:

Lemma 5.1. *Let T_1 and T_2 be two complete subdesigns of T. Then $T' = \langle B(T_1) \cap B(T_2), \Pi(T_2) \cap \Pi(T_2)\rangle$ is either a trivial pair with no blocks and less than t points, or a complete subdesign of T.*

Proof: If $\Pi = \Pi(T_1) \cap \Pi(T_2)$ has t or more points, there is a t-set $A \subseteq \Pi(T_1) \cap \Pi(T_2)$. Since T_1 and T_2 are both complete, the $\lambda(T)$ blocks of T containing A are all in both T_1 and T_2, so there are $\lambda(T)$ blocks in $B(T_1) \cap B(T_2) = B$. This also holds for any t-set in Π, so T' is a complete subdesign.

By simple induction and the associativity of intersections, Lemma 5.1 can be extended to any finite number of complete subdesigns, so for any set of complete subdesigns $T_1, T_2, \ldots, T_i$, $\bigcap_i T_i = \langle \bigcap_i B(T_i), \bigcap_i \Pi(T_i)\rangle$, is always either a complete subdesign or no blocks and less than t points.

DEFINITION 5.2. For any t-set $A \subseteq \Pi(T)$, let $T_1(A), T_2(A), \ldots, T_n(A)$ be the complete subdesigns of T such that $A \subseteq \Pi(T_i(A))$. Then the *subdesign generated* by A is written $T(A)$ and defined to be $\bigcap_i T_i(A)$.

This is, of course, always a complete subdesign, because

$$A \subseteq \bigcap_i \Pi(T_1(A)),$$

which then has at least t-points. Returning now to the special case of a t-design with t-ply homogeneous group, the following results show the importance of the subdesigns of the form $T(A)$.

PROPOSITION 5.3. For any $\alpha \in G$, $[T(A)]^\alpha = T(A^\alpha)$. In particular, $T(A)$ is isomorphic to $T(B)$ for any t-sets $A, B \subseteq \Pi(T)$.

PROOF: Alpha sends every complete subdesign containing A into one containing A^α, hence $[T(A)]^\alpha \supseteq T(A^\alpha)$. By the same argument on $T(A^\alpha)$, $[T(A^\alpha)]^{\alpha^{-1}} \supseteq T(A)$, or $T(A^\alpha) \supseteq [T(A)]^\alpha$, hence $T(A^\alpha) = [T(A)]^\alpha$. To show the isomorphism of $T(A)$ and $T(B)$, let β be any automorphism sending A into B. Then $[T(A)]^\beta = T(A^\beta) = T(B)$, and β is the required isomorphism.

THEOREM 5.4. *Let T_1 be any complete subdesign of T (or T itself). Then for any t-set $A \subseteq \Pi(T_1)$, $T(A) \lhd T_1$.*

PROOF: We shall show that the various distinct subdesigns $T(A_i)$ (A_i a t-set in T_1) are codesigns whose block sets partition $B(T_1)$. From Proposition 5.3, they are all isomorphic, hence they are codesigns. Their block sets obviously exhaust $B(T_1)$. Suppose now that $B(T(A))$ and $B(T(B))$ have a block Φ in common. But then there would be a t-set $C \subseteq \Phi$, and since $T(A)$ and $T(B)$ are complete, by definition $T(C) \subseteq T(A) \cap T(B)$. Since $T(C)$ is isomorphic to $T(A)$, also to $T(B)$, the only possible conclusion is $T(B) = T(C) = T(A)$, and so if $T(A) \neq T(B)$, $B(T(A)) \cap B(T(B)) = \varnothing$, and the block sets partition $B(T_1)$, and all subdesigns $T(A)$ are normal in T_1.

Note that, if $\lambda(T) = 1$, $T(A) = \langle\{\Phi\}, \Phi\rangle$, where Φ is the single block containing A. For $\lambda(T) > 1$, $T(A)$ must be non-trivial, and we have

THEOREM 5.5. *If $\lambda(T) > 1$, T is simple if and only if T contains no non-trivial complete subdesigns.*

PROOF: Every normal subdesign is complete, so the condition is sufficient. Conversely, suppose T contains a non-trivial proper complete subdesign T_1. Then for any t-set $A \subseteq \Pi(T_1)$, $T(A) \subseteq T_1 \subseteq T$, so by Theorem 5.4, $T(A)$ is a non-trivial normal subdesign of T.

THEOREM 5.6. *Each $T(A)$ is simple.*

PROOF: Suppose $T_1 \lhd T(A)$. Then $\lambda(T_1) = \lambda(T(A)) = \lambda(T)$. If B is any t-set in $\Pi(T_1)$, $T(B) \subseteq T_1$ by definition, and from Theorem 5.4, $T(B) \lhd T_1 \lhd T(A)$. Since $T(B)$ is isomorphic to $T(A)$ from Proposition 5.3, we can only have $T(B) = T_1 = T(A)$. Therefore every normal subdesign of $T(A)$ is $T(A)$, and $T(A)$ is simple.

THEOREM 5.7. *If $\lambda(T) > 1$, every simple complete subdesign T_1 of T is a $T(A)$ and so is normal in T. In particular, if T is simple, $T = T(A)$ for every t-set $A \subseteq \Pi(T)$.*

PROOF: Let A_1 be a t-set in $\Pi(T_1)$. From Theorem 5.4 applied to T_1, $T(A_1) \lhd T_1$, and since $\lambda(T(A_1)) = \lambda(T_1) = \lambda(T) > 1$, $T(A_1)$ is a non-trivial normal subdesign of T_1, hence the simplicity of T_1 implies that $T_1 = T(A_1)$.

Given a design T, the subdesigns of the form $T(A)$ are relatively simple to construct: one simply takes all the blocks containing A and all the points on those blocks, then continues the same process, using new t-sets from the new blocks to produce newer blocks, until each t-set in the points produced already occurs $\lambda(T)$ times in the blocks already found. However, the subdesigns $T(A)$ can be found directly from examination of the automorphism group:

THEOREM 5.8. *If T_1 is a simple complete subdesign of T,*

(1) *$G_{\Pi(T_1)}$ is t-ply homogeneous on $\Pi(T_1)$,*

(2) *$G_\Phi \subseteq G_{\Pi(T_1)}$ for any $\Phi \in B(T_1)$, and*

(3) *$G_A \subseteq G_{\Pi(T_1)}$ for any t-set $A \subseteq \Pi(T_1)$.*

PROOF: T_1 is equal to $T(A)$ for any t-set $A \subseteq \Pi(T_1)$. Therefore any automorphism which sends a t-set $A \subseteq \Pi(T_1)$ into another t-set $B \subseteq \Pi(T_1)$ also sends $T_1 = T(A)$ into $T(B) = T_1$, i.e., fixes T_1.

(1) Let A, B be two t-sets in $\Pi(T_1)$. There is an $\alpha \in G$ such that $A^\alpha = B$. By the above argument $T_1^\alpha = T_1$, so $\alpha \in G_{\Pi(T_1)}$. Since A, B were chosen arbitrarily, $G_{\Pi(T_1)}$ must be t-ply homogeneous on $\Pi(T_1)$.

(2) Let $\alpha \in G_\Phi$ be arbitrary. Alpha then sends any t-set of Φ into another t-set of Φ. Since $\Phi \subseteq \Pi(T_1)$, these two t-sets are in $\Pi(T_1)$, so the above

argument applies to α here, so $\alpha \in G_{\Pi(T_1)}$. Since α was chosen arbitrarily, $G_\Phi \subseteq G_{\Pi(T_1)}$.

(3) Any $\alpha \in G_A$ sends A into itself, so again by the above argument, $\alpha \in G_{\Pi(T_1)}$.

THEOREM 5.9. *Let T_1 be a simple complete subdesign of T. Then, for each block Φ of T_1, $T_1 = \Phi^{G_{\Pi(T_1)}}$.*

PROOF: From Theorem 5.8, $H = G_{\Pi(T_1)}$ is t-ply homogeneous on $\Pi(T_1)$, hence, by Theorem 2.1, $T' = \Phi^H$ is a t-design and a subdesign of T. We now calculate the parameters of T' from Theorem 2.1.

$v(T') = v(T_1)$, since H is transitive on $\Pi(T_1)$.

$k(T') = k(T_1)$ by definition.

$\lambda(T') = \binom{k}{t} \dfrac{|H_A|}{|H_\Phi|}$ where A is any t-set in $\Pi(T_1)$.

But, from Theorem 5.8, $H_A = G_A$ and $H_\Phi = G_\Phi$, hence

$$\lambda(T') = \binom{k}{t} \frac{|G_A|}{|G_\Phi|} = \lambda(T) = \lambda(T_1).$$

T' is then a subdesign of T_1 with the same parameters, hence must be all of T_1.

These results lead to a converse of Theorem 5.8 and a characterization of simple t-designs purely in terms of their automorphism groups:

THEOREM 5.10. *Let A be any t-set of $\Pi(T)$, let Φ be any block of T containing A, and suppose $\lambda(T) > 1$. Then T is simple if and only if G contains no proper subgroup H such that*

(1) $G_A \subseteq H$,

(2) $G_\Phi \subseteq H$,

(3) *H is t-ply homogeneous on the points in the t-sets A^H.*

PROOF: If T has a proper normal subdesign T^*, then T^* has a codesign T' containing Φ and so A. Since T' is also normal, it is complete, therefore $T(A) \subseteq T'$ is a proper simple complete subdesign of T, and from Theorem 5.8, $H = G_{\Pi(T(A))}$ satisfies (1), (2), and (3). Conversely, suppose such an H exists. Then Φ^H is a subdesign by Theorem 2.1.

$$\lambda(\Phi^H) = \binom{k}{t} \frac{|H_A|}{|H_\Phi|} = \binom{k}{t} \frac{|G_A|}{|G_\Phi|} = \lambda(T)$$

as before, so Φ^H is complete. But then $T(A) \subsetneq \Phi^H$ is a proper non-trivial normal subdesign.

We can also relate the quotient designs of T to the group G:

THEOREM 5.11. *If T_1 is a simple normal subdesign of T, then*

$$T/T_1 = [\Pi(T_1)]^G.$$

PROOF: $T_1 = T(A_1)$ for any t-set A_1 in $\Pi(T_1)$; furthermore, all codesigns of T_1 must also be of the form $T(A_i)$, $1 \leqslant i \leqslant m$. Therefore,

$$T/T_1 = \langle\{\Pi(T(A_i))\}_{i=1}^m, \quad \Pi(T)\rangle.$$

But, from Proposition 5.3, for every $\alpha \in G$ and every i, $[T(A_i)]^\alpha = T(A_i^\alpha)$, so $[\Pi(T(A_i))]^\alpha = \Pi(T(A_i^\alpha))$, and G permutes the sets $\Pi(T(A_i))$. Furthermore, G is t-ply homogeneous, hence for each i there is an $\alpha_i \in G$ such that $A_1^{\alpha_i} = A_i$, so $[\Pi(T(A_1))]^{\alpha_i} = \Pi(T(A_i))$. Thus the sets $\{[\Pi(T_1)]^\alpha\}_{\alpha\in G}$ are exactly the sets $\{\Pi(T(A_i))\}_{i=1}^m$, and $T/T_1 = [\Pi(T_1)]^G$.

THEOREM 5.12. *Let T_1 be a simple complete subdesign of T. Then*

$$G(T) \subsetneq G(T/T_1).$$

PROOF: $T_1 = T(A)$ for some t-set A. By Proposition 5.3, any automorphism of T permutes the various sets $T(A_i)$, hence from Theorem 5.11, every automorphism of T permutes the blocks of T/T_1.

Note that this inequality may indeed be strict, because a permutation on $\Pi(T)$ could permute the sets $\Pi(T(A_i))$, but not the actual blocks of the subdesigns $T(A_i)$. However, we do have that the group of T/T_1 is t-ply homogeneous, hence the previous analysis applies to T/T_1 and $G(T/T_1)$, and, by continuation, the complete normal structure of T may be derived from the groups of the various normal and quotient designs.

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